Textbook Questions: Chapter 17 Multivariable Calculus

17.1 Multivariable Functions and Partial Derivatives (Q. 1-29, 33)

PROBLEMS 17.1

In Problems 1–26, a function of two or more variables is given. Find the partial derivative of the function with respect to each of the variables.

1. $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$

2. $f(x, y) = 2x^2 + 3xy$

3. $f(x, y) = 2y + 1$

4. $f(x, y) = e^{\pi} \ln 2$

5. $g(x, y) = 3x^4y + 2xy^2 - 5xy + 8x - 9y$

6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$

7. $g(p, q) = \sqrt{pq}$

8. $g(w, z) = \sqrt[3]{w^2 + z^2}$

9. $h(s, t) = \dfrac{s^2 + 1}{t^2 - 1}$

10. $h(u, v) = \dfrac{8uv^2}{u^2 + v^2}$

11. $u(q_1, q_2) = \ln \sqrt{q_1 + 2} + \ln \sqrt[3]{q_2 + 5}$

12. $Q(l, k) = 2l^{0.38}k^{1.79} - 3l^{1.03} + 2k^{0.13}$

13. $h(x, y) = \dfrac{x^2 + 3xy + y^2}{\sqrt{x^2 + y^2}}$

14. $h(x, y) = \dfrac{x + 4}{xy^2 - x^2y}$

15. $z = e^{5xy}$

16. $z = (x^3 + y^3)e^{xy+3x+3y}$

17. $z = 5x \ln(x^2 + y)$

18. $z = \ln(5x^3y^2 + 2y^4)^4$

19. $f(r, s) = \sqrt{r - s}(r^2 - 2rs + s^2)$

20. $f(r, s) = \sqrt{rs}\, e^{2+r}$

21. $f(r, s) = e^{3-r} \ln(7 - s)$

22. $f(r, s) = (5r^2 + 3s^3)(2r - 5s)$

23. $g(x, y, z) = 2x^3y^2 + 2xy^3z + 4z^2$

24. $g(x, y, z) = xy^2z^3 + x^3yz^2 + x^2y^3z$

25. $g(r, s, t) = e^{s+t}(r^2 + 7s^3)$

26. $g(r, s, t, u) = rs \ln(t)e^u$

In Problems 27–34, evaluate the given partial derivatives.

27. $f(x, y) = x^3y + 7x^2y^2$; $f_x(1, -2)$

28. $z = \sqrt{2x^3 + 5xy + 2y^2}$; $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}$

29. $g(x, y, z) = e^{x+y+z}\sqrt{x^2 + y^2 + z^2}$; $g_z(0, 3, 4)$

30. $g(x, y, z) = \dfrac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$, $g_y(1, 1, 5)$

31. $h(r, s, t, u) = (rst^2u) \ln(1 + rstu)$; $h_t(1, 1, 0, 1)$

32. $h(r, s, t, u) = \dfrac{7r + 3s^2u^2}{s}$; $h_t(4, 3, 2, 1)$

33. $f(r, s, t) = rst(r^2 + s^3 + t^4)$; $f_s(1, -1, 2)$

34. $z = \dfrac{x^2 - y^2}{e^{x^2-y^2}}$; $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}}$

35. If $z = xe^{x-y} + ye^{y-x}$, show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + e^{y-x}$$

36. Stock Prices of a Dividend Cycle In a discussion of stock prices of a dividend cycle, Palmon and Yaari[1] consider the function f given by

$$u = f(t, r, z) = \frac{(1 + r)^{1-z} \ln(1 + r)}{(1 + r)^{1-z} - t}$$

where u is the instantaneous rate of ask-price appreciation, r is an annual opportunity rate of return, z is the fraction of a dividend cycle over which a share of stock is held by a midcycle seller, and t is the effective rate of capital gains tax. They claim that

$$\frac{\partial u}{\partial z} = \frac{t(1 + r)^{1-z} \ln^2(1 + r)}{[(1 + r)^{1-z} - t]^2}$$

Verify this.

37. Money Demand In a discussion of inventory theory of money demand, Swanson[2] considers the function

$$F(b, C, T, i) = \frac{bT}{C} + \frac{iC}{2}$$

and determines that $\dfrac{\partial F}{\partial C} = -\dfrac{bT}{C^2} + \dfrac{i}{2}$. Verify this partial derivative.

38. Interest Rate Deregulation In an article on interest rate deregulation, Christofi and Agapos[3] arrive at the equation

$$r_L = r + D\frac{\partial r}{\partial D} + \frac{dC}{dD} \qquad (3)$$

where r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, and D is the savings deposit level. Christofi and Agapos state that

$$r_L = r\left[\frac{1 + \eta}{\eta}\right] + \frac{dC}{dD} \qquad (4)$$

where $\eta = \dfrac{r/D}{\partial r/\partial D}$ is the deposit elasticity with respect to the deposit rate. Express Equation (3) in terms of η to verify Equation (4).

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[1] D. Palmon and U. Yaari, "Taxation of Capital Gains and the Behavior of Stock Prices over the Dividend Cycle," *The American Economist*, XXVII, no. 1 (1983), 13–22.

[2] P. E. Swanson, "Integer Constraints on the Inventory Theory of Money Demand," *Quarterly Journal of Business and Economics*, 23, no. 1 (1984), 32–37.

[3] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research*, XX (1984), 39–49.

17.2 Applications of Partial Derivatives (Q. 1-5, 13)

PROBLEMS 17.2

For the joint-cost functions in Problems 1–3, find the indicated marginal cost at the given production level.

1. $c = 7x + 0.3y^2 + 2y + 900$; $\frac{\partial c}{\partial y}, x = 20, y = 30$

2. $c = 2x\sqrt{x + y} + 6000$; $\frac{\partial c}{\partial x}, x = 70, y = 74$

3. $c = 0.03(x + y)^3 - 0.6(x + y)^2 + 9.5(x + y) + 7700$; $\frac{\partial c}{\partial x}, x = 50, y = 80$

For the production functions in Problems 4 and 5, find the marginal productivity functions $\partial P/\partial k$ and $\partial P/\partial l$.

4. $P = 15lk - 3l^2 + 5k^2 + 500$

5. $P = 2.527l^{0.314}k^{0.686}$

6. Cobb–Douglas Production Function In economics, a Cobb–Douglas production function is a production function of the form $P = Al^{\alpha}k^{\beta}$, where A, α, and β are constants and $\alpha + \beta = 1$. For such a function, show that

(a) $\partial P/\partial l = \alpha P/l$ **(b)** $\partial P/\partial k = \beta P/k$

(c) $l\frac{\partial P}{\partial l} + k\frac{\partial P}{\partial k} = P$. This means that summing the products of the marginal productivity of each factor and the amount of that factor results in the total product P.

In Problems 7–9, q_A and q_B are demand functions for products A and B, respectively. In each case, find $\partial q_A/\partial p_A$, $\partial q_A/\partial p_B$, $\partial q_B/\partial p_A$, and $\partial q_B/\partial p_B$, and determine whether A and B are competitive, complementary, or neither.

7. $q_A = 1500 - 40p_A + 3p_B$; $q_B = 900 + 5p_A - 20p_B$

8. $q_A = 20 - p_A - 2p_B$; $q_B = 50 - 2p_A - 3p_B$

9. $q_A = \frac{100}{p_A\sqrt{p_B}}$; $q_B = \frac{500}{p_B\sqrt[3]{p_A}}$

10. Canadian Manufacturing The production function for the Canadian manufacturing industries for 1927 is estimated by[6] $P = 33.0l^{0.46}k^{0.52}$, where P is product, l is labor, and k is capital. Find the marginal productivities for labor and capital, and evaluate when $l = 1$ and $k = 1$.

11. Dairy Farming An estimate of the production function for dairy farming in Iowa (1939) is given by[7]

$$P = A^{0.27}B^{0.01}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$$

where P is product, A is land, B is labor, C is improvements, D is liquid assets, E is working assets, and F is cash operating expenses. Find the marginal productivities for labor and improvements.

12. Production Function Suppose a production function is given by $P = \frac{kl}{3k + 5l}$.

(a) Determine the marginal productivity functions.

(b) Show that when $k = l$, the marginal productivities sum to $\frac{1}{8}$.

13. MBA Compensation In a study of success among graduates with master of business administration (MBA) degrees, it was estimated that for staff managers (which include accountants, analysts, etc.), current annual compensation (in dollars) was given by

$$z = 43{,}960 + 4480x + 3492y$$

where x and y are the number of years of work experience before and after receiving the MBA degree, respectively.[8] Find $\partial z/\partial x$ and interpret your result.

14. Status A person's general status S_g is believed to be a function of status attributable to education, S_e, and status attributable to income, S_i, where S_g, S_e, and S_i are represented numerically. If

$$S_g = 7\sqrt[3]{S_e}\sqrt{S_i}$$

determine $\partial S_g/\partial S_e$ and $\partial S_g/\partial S_i$ when $S_e = 125$ and $S_i = 100$, and interpret your results.[9]

15. Reading Ease Sometimes we want to evaluate the degree of readability of a piece of writing. Rudolf Flesch[10] developed a function of two variables that will do this, namely,

$$R = f(w, s) = 206.835 - (1.015w + 0.846s)$$

where R is called the *reading ease score*, w is the average number of words per sentence in 100-word samples, and s is the average number of syllables in such samples. Flesch says that an article for which $R = 0$ is "practically unreadable," but one with $R = 100$ is "easy for any literate person." **(a)** Find $\partial R/\partial w$ and $\partial R/\partial s$. **(b)** Which is "easier" to read: an article for which $w = w_0$ and $s = s_0$, or one for which $w = w_0 + 1$ and $s = s_0$?

17.3 Higher Order Partial Derivatives (Q. 1-9, 16, 19, 21)

PROBLEMS 17.3

In Problems 1–10, find the indicated partial derivatives.

1. $f(x,y) = 5x^3y$; $f_x(x,y), f_{xy}(x,y), f_{yx}(x,y)$
2. $f(x,y) = 2x^3y^2 + 6x^2y^3 - 3xy$; $f_x(x,y),\ f_{xx}(x,y)$
3. $f(x,y) = 7x^2 + 3y$; $f_y(x,y),\ f_{yy}(x,y),\ f_{yyx}(x,y)$
4. $f(x,y) = (x^2 + xy + y^2)(xy + x + y)$; $f_x(x,y),\ f_{xy}(x,y)$
5. $f(x,y) = 9e^{2xy}$; $f_y(x,y),\ f_{yx}(x,y),\ f_{yxy}(x,y)$
6. $f(x,y) = \ln(x^2 + y^3) + 5$; $f_x(x,y), f_{xx}(x,y), f_{xy}(x,y)$
7. $f(x,y) = (x+y)^2(xy)$; $f_x(x,y),\ f_y(x,y),\ f_{xx}(x,y),\ f_{yy}(x,y)$
8. $f(x,y,z) = x^2y^3z^4$; $f_x(x,y,z),\ f_{xz}(x,y,z),\ f_{zx}(x,y,z)$
9. $z = \ln\sqrt{x^2+y^2}$; $\dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial y^2}$
10. $z = \dfrac{\ln(x^2+5)}{y}$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial y\,\partial x}$

In Problems 11–16, find the indicated value.

11. If $f(x,y,z) = 5$, find $f_{yxxz}(4,3,-2)$.
12. If $f(x,y,z) = z^2(3x^2 - 4xy^3)$, find $f_{xyz}(1,2,3)$.

13. If $f(l,k) = 3l^3k^6 - 2l^2k^7$, find $f_{klk}(2,1)$.
14. If $f(x,y) = x^3y^2 + x^2y - x^2y^2$, find $f_{xxy}(2,3)$ and $f_{xyx}(2,3)$.
15. If $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$.
16. If $f(x,y) = 2x^3 + 3x^2y + 5xy^2 + 7y^3$, find $f_{xy}(2,3)$.
17. **Cost Function** Suppose the cost, c, of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Find the value of

$$\frac{\partial^2 c}{\partial q_A\,\partial q_B}$$

when $p_A = 25$ and $p_B = 4$.

18. For $f(x,y) = x^4y^4 + 3x^3y^2 - 7x + 4$, show that

$$f_{xyx}(x,y) = f_{xxy}(x,y)$$

19. For $f(x,y) = e^{x^2+xy+y^2}$, show that

$$f_{xy}(x,y) = f_{yx}(x,y)$$

20. For $f(x,y) = e^{xy}$, show that

$$f_{xx}(x,y) + f_{xy}(x,y) + f_{yx}(x,y) + f_{yy}(x,y) = f(x,y)((x+y)^2 + 2)$$

21. For $w = \ln(x^2+y^2)$, show that $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$. For $w = \ln(x^2+y^2+z^2)$, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2}{x^2+y^2+z^2}$$

17.4 Maxima and Minima for Functions of two variables (Q. 1-23, 29)

PROBLEMS 17.4

In Problems 1–6, find the critical points of the functions.

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

2. $f(x, y) = x^2 + 3y^2 - 4x - 30y$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 4y^2 - 6x - 32y + 1$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

Find the selling prices p_A and p_B that maximize the company's profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

$$p = 92 - q_A - q_B$$

where q_A and q_B denote the output produced and sold by A and B, respectively. For A, the cost function is $c_A = 10q_A$; for B, it is $c_B = 0.5q_B^2$. Suppose the firms decide to enter into an agreement on output and price control by jointly acting as a monopoly. In this case, we say they enter into *collusion*. Show that the profit function for the monopoly is given by

$$P = pq_A - c_A + pq_B - c_B$$

Express P as a function of q_A and q_B, and determine how output should be allocated so as to maximize the profit of the monopoly.

31. Suppose $f(x, y) = x^2 + 3y^2 + 9$, where x and y must satisfy the equation $x + y = 2$. Find the relative extrema of f, subject to the given condition on x and y, by first solving the second equation for y (or x). Substitute the result in the first equation. Thus, f is expressed as a function of one variable. Now find where relative extrema for f occur.

32. Repeat Problem 31 if $f(x, y) = x^2 + 4y^2 + 11$, subject to the condition that $x - y = 1$.

33. Suppose the joint-cost function

$$c = q_A^2 + 3q_B^2 + 2q_Aq_B + aq_A + bq_B + d$$

has a relative minimum value of 15 when $q_A = 3$ and $q_B = 1$. Determine the values of the constants a, b, and d.

34. Suppose that the function $f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$ and suppose that $D(a, b) > 0$.

(a) Show that $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.
(b) Show that $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. Profit from Competitive Products A monopolist sells two competitive products, A and B, for which the demand equations are

$$p_A = 35 - 2q_A^2 + q_B$$

and

$$p_B = 20 - q_B + q_A$$

The joint-cost function is

$$c = -8 - 2q_A^3 + 3q_Aq_B + 30q_A + 12q_B + \frac{1}{2}q_A^2$$

(a) How many units of A and B should be sold to obtain a relative maximum profit for the monopolist? Use the second-derivative test to justify your answer.
(b) Determine the selling prices required to realize the relative maximum profit. Also, find this relative maximum profit.

36. Profit and Advertising A retailer has determined that the number of TV sets he can sell per week is

$$\frac{7x}{2 + x} + \frac{4y}{5 + y}$$

where x and y represent his weekly expenditures (in dollars) on newspaper and radio advertising, respectively. The profit is \$300 per sale, less the cost of advertising, so the weekly profit is given by the formula

$$P = 300\left(\frac{7x}{2 + x} + \frac{4y}{5 + y}\right) - x - y$$

Find the values of x and y for which the profit is a relative maximum. Use the second-derivative test to verify that your answer corresponds to a relative maximum profit.

37. Profit from Tomato Crop The revenue (in dollars per square meter of ground) obtained from the sale of a crop of tomatoes grown in an artificially heated greenhouse is given by

$$r = 5T(1 - e^{-x})$$

where T is the temperature (in °C) maintained in the greenhouse and x is the amount of fertilizer applied per square meter. The cost of fertilizer is $20x$ dollars per square meter, and the cost of heating is given by $0.1T^2$ dollars per square meter.

(a) Find an expression, in terms of T and x, for the profit per square meter obtained from the sale of the crop of tomatoes.
(b) Verify that the pairs

$$(T, x) = (20, \ln 5) \quad \text{and} \quad (T, x) = (5, \ln \tfrac{5}{4})$$

are critical points of the profit function in part (a). (*Note:* You need not derive the pairs.)
(c) The points in part (b) are the only critical points of the profit function in part (a). Use the second-derivative test to determine whether either of these points corresponds to a relative maximum profit per square meter.

17.5 Lagrange Multipliers (Q. 1-7, 13,15)

PROBLEMS 17.5

In Problems 1–12, find, by the method of Lagrange multipliers, the critical points of the functions, subject to the given constraints.

1. $f(x, y) = x^2 + 4y^2 + 6;\quad 2x - 8y = 20$

2. $f(x, y) = 3x^2 - 2y^2 + 9;\quad x + y = 1$

3. $f(x, y, z) = x^2 + y^2 + z^2;\quad x + y + z = 1$

4. $f(x, y, z) = x + y + z;\quad xyz = 8$

5. $f(x, y, z) = 2x^2 + xy + y^2 + z;\quad x + 2y + 4z = 3$

6. $f(x, y, z) = xyz^2;\quad x - y + z = 20\ (xyz^2 \neq 0)$

7. $f(x, y, z) = xyz;\quad x + y + z = 1\ (xyz \neq 0)$

8. $f(x, y, z) = x^2 + 4y^2 + 9z^2;\quad x + y + z = 3$

9. $f(x, y, z) = x^2 + 2y - z^2;\quad 2x - y = 0,\ y + z = 0$

10. $f(x, y, z) = x^2 + y^2 + z^2;\quad x + y + z = 4,\ x - y + z = 4$

11. $f(x, y, z) = xy^2z;\quad x + y + z = 1,\ x - y + z = 0\ (xyz \neq 0)$

12. $f(x, y, z, w) = x^2 + 2y^2 + 3z^2 - w^2;\ 4x + 3y + 2z + w = 10$

13. Production Allocation To fill an order for 100 units of its product, a firm wishes to distribute production between its two plants, plant 1 and plant 2. The total-cost function is given by

$$c = f(q_1, q_2) = q_1^2 + 3q_1 + 25q_2 + 1000$$

where q_1 and q_2 are the numbers of units produced at plants 1 and 2, respectively. How should the output be distributed in order to minimize costs? (Assume that the critical point obtained corresponds to the minimum cost.)

14. Production Allocation Repeat Problem 13 if the cost function is

$$c = 3q_1^2 + q_1q_2 + 2q_2^2$$

and a total of 200 units are to be produced.

15. Maximizing Output The production function for a firm is

$$f(l, k) = 12l + 20k - l^2 - 2k^2$$

The cost to the firm of l and k is 4 and 8 per unit, respectively. If the firm wants the total cost of input to be 88, find the greatest output possible, subject to this budget constraint. (You may assume that the critical point obtained does correspond to the maximum output.)